

Properties of Zero-Lag Long-Range Synchronization via Dynamical Relaying

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In a recent letter, Fisher *et al.* reported the phenomenon of zero-lag long range isochronous synchronization via dynamical relaying in systems with delay [Phys. Rev. Lett. **97**, 123902 (2006)]. They reported that when one has two coupled systems A and C, with delay between them, then the introduction of a third element B between A and C will allow them to synchronize even in regions of the parameter space where this was not possible without the presence of B. Here we study in detail the phenomenon and verify that in all the cases studied (including the ones reported by Fisher *et al.*) this occurs due to the tendency of A and B and B and C to be in antiphase synchronization and if A is in antiphase with B and B is in antiphase with C, it will imply that A and C are inphase. We show this in coupled quadratic maps, Kuramoto and Rössler oscillators. We also report that there is a simpler configuration where the same phenomenon occurs and has nearly the same features of the cases studied by Fisher *et al.*

INTRODUCTION

Synchronization phenomenon is widely observed in nature and in artificial systems. The first description of synchronization is believed to have been made by Huygens. He observed that two pendulum clocks suspended in the same wooden beam tend to synchronize in opposite swings[1]. In more recent studies, synchronization has been observed in fireflies, pacemaker cells, networks of neurons, etc[2]. In brain activity, it has been found that cognitive acts, such as face perception, is associated with zero-lag (i.e., isochronal) synchronization of gamma oscillations[3]. Other studies have shown neural synchronization in widely separated areas of cortical regions [4, 5].

In a recent publication, Fisher *et al.*[6] reported the phenomenon of zero-lag long range synchronization via dynamical relaying. That is, when one has two coupled systems, with delay between them, introducing a third one will allow synchronization between the two outer elements in regions of the parameter space where this was impossible without the middle element. In their letter they showed that this occurs for coupled lasers (in experimental and numerical studies), coupled neurons and stated that they observed the same phenomenon in a large variety of dynamical systems (such as excitable systems, oscillators and maps).

The aim of this paper is to provide an explanation for this phenomenon. We find that this occurs because two identical systems lose synchronization as the delay between them is increased in such a way that they get in antiphase synchronization. That is, in the case of oscillators their phase separation is π . When a third element is added between those elements, then the middle element will be in antiphase with the outer elements and therefore the outer elements will be inphase. Here we call the attention to the fact that it is not any finite delay that will cause loss of synchronization between two coupled systems, as we will see below. For loss of synchronization to occur the delay has to be in a given range of values

and therefore this phenomenon is only important in that region.

In their letter, Fisher *et al.* studied bidirectional coupling between the middle and outer elements. Here we show that nearly identical results can be obtained in a simpler configuration, that is, the subsystem coupling as studied in [7]. We demonstrate our results in quadratic maps[8], Kuramoto[9] and coupled Rössler[10] oscillators. In the case of Kuramoto oscillators, we show analytically that when all the oscillators are identical, synchronization between the two outer elements always occur, independently of the delay and coupling values. Although the synchronization properties do not depend on if the coupling is bidirectional or of the subsystem type, in both cases it is necessary that the parameters of the outer elements be the same.

The paper is organized as follows: In the next section we study zero-lag synchronization via dynamical relaying in a discrete time system, that is, the quadratic map. Then we turn our attention to this phenomenon in continuous time systems. In particular, we study the Kuramoto and Rössler oscillators. The last section is dedicated to discussion.

QUADRATIC MAPS

Here we study a system of coupled quadratic maps. For two coupled quadratic maps with delay of two time steps, we have

$$\begin{aligned} x_1^{t+1} &= (1-k)f(x_1^t) + kf(x_2^{t-2}) \\ x_2^{t+1} &= (1-k)f(x_2^t) + kf(x_1^{t-2}), \end{aligned} \quad (1)$$

where $f = 1 - ax^2$, $a \in [0, 2]$, with f mapping the interval $[-1, 1]$ into itself. The case when the coupling strength, k , is zero has been extensively studied in the literature and one knows that the logistic map presents period doubling bifurcations, leading to a chaotic behavior as a is increased[8]. In the coupling scheme studied in [6] our

system would be

$$\begin{aligned} x_1^{t+1} &= (1-k)f(x_1^t) + kf(x_2^{t-1}) \\ x_2^{t+1} &= (1-k)f(x_2^t) + k[f(x_1^{t-1}) + f(x_3^{t-1})]/2. \\ x_3^{t+1} &= (1-k)f(x_3^t) + kf(x_2^{t-1}). \end{aligned} \quad (2)$$

Note that in Eq. 1 the delay is of two time steps and in Eq. 2 the delay is of one time step, since, as done by Fisher *et al.*, one assumes that the delay between elements 1 and 2, as well as between 2 and 3 is half of the delay between 1 and 3.

For the system of coupled quadratic maps considered above, we noticed that more than one basin of attraction is present. Therefore, studying the region of synchronization for a given initial condition, may not give a full picture of the synchronization region. We observed that, as in the case without delay[7], the transverse Liapunov exponent correctly determines the region where synchronization is possible between two elements of a given system. The region of synchronization found in this way does not depend on the initial condition used, since by construction the transverse Liapunov exponent measures the separation (or shrinkage) of trajectories that are started nearby to each other. We have adapted the method of Benettin *et al.*[11] to study the transverse Liapunov exponent of the coupled quadratic maps. If we are studying the synchronization between element a and b , after the transient dies out, we evolve the orbit of b by making it slightly different from the orbit of element a . Then we verify how the orbits of a and b differ after one iteration. The perturbation is renormalized in the direction of maximum growth and the process is repeated many times. The transverse Liapunov exponent is given by the average logarithm (in this paper we use base 2) of the growth (or shrinkage) of the perturbation along the orbit. In Figs. 1(a) and (b) we show the region where the transverse Liapunov is non negative (shaded area) for, respectively, elements 1 and 2 in Eq. 1 and elements 1 and 3 in Eq. 2. We see that the region where synchronization is possible (i.e., when the transverse Liapunov exponent is negative) is greatly increased with the scheme given by Eq. 2. Although the study of synchronization between 1 and 2 in Eq. 2 is not the main goal of this work, we show in Fig. 1(c) for the sake of comparison, the region of non negative transverse Liapunov exponent for their synchronization orbit. We see that synchronization between 1 and 2 in Eq. 2 occurs in a smaller region of the parameter space, when we compare with synchronization between 1 and 2 in Eq. 1 and between 1 and 3 in Eq. 2. This shows that 1 and 3 can be synchronized even when 1 and 2 cannot.

Our next step is to compare these results with the synchronization properties of 1 and 3 when the system is coupled via a subsystem coupling, as studied in [7] for systems without delay in the context of chaotic synchronization. In this situation, the equations that govern the

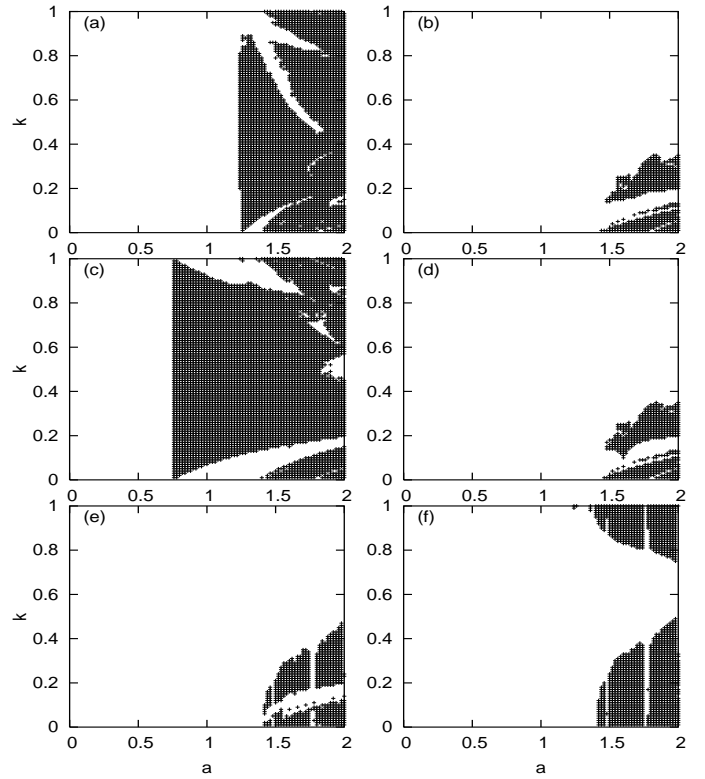


FIG. 1: Region of nonnegative transverse Liapunov exponent for synchronization (shaded) between (a) 1 and 2 in Eq. 1, (b) 1 and 3 in Eq. 2, (c) 1 and 2 in Eq. 2, (d) 1 and 3 in Eq. 3, (e) 1 and 3 for subsystem coupling without delay and (f) 1 and 2 for subsystem coupling without delay. The initial conditions used were for Eq. 1: $x_1^{-2} = 0.3$, $x_2^{-2} = 0.1$, $x_1^{-1} = 0.25$, $x_2^{-1} = 0.05$, $x_1^0 = 0.2$ and $x_2^0 = 0.0$. For Eqs. 2 and Eqs. 3 we used: $x_1^{-1} = 0.25$, $x_2^{-1} = 0.05$, $x_3^{-1} = 0.55$, $x_1^0 = 0.2$, $x_2^0 = 0.0$ and $x_3^0 = 0.5$. For the subsystem coupling without delay we used: $x_1^0 = 0.2$, $x_2^0 = 0.0$, $x_3^0 = 0.5$. A transient of 20000 iterations was discarded and the Liapunov exponent was considered non negative if it was greater than 10^{-4} after 50000 iterations.

dynamics of the coupled system are

$$\begin{aligned} x_1^{t+1} &= (1-k)f(x_1^t) + kf(x_2^{t-1}) \\ x_2^{t+1} &= (1-k)f(x_2^t) + kf(x_1^{t-1}) \\ x_3^{t+1} &= (1-k)f(x_3^t) + kf(x_2^{t-1}). \end{aligned} \quad (3)$$

We show in Fig. 1(d) the region of non negative transverse Liapunov exponent (shaded area) for the orbits of elements 1 and 3. When compared with Fig. 1(b) one sees that the two shaded regions are nearly identical to each other, showing that there is basically no difference of the parameter region where synchronization can occur between 1 and 3 in one coupling versus the other. We have noticed only very minor differences in the parameter region of synchronization between 1 and 2 in Eq. 3 when compared with Fig. 1(c). For the sake of completeness, we also show in Figs. 1(e) and (f) the region of non negative Liapunov exponent between 1 and 3 and 1 and 2,

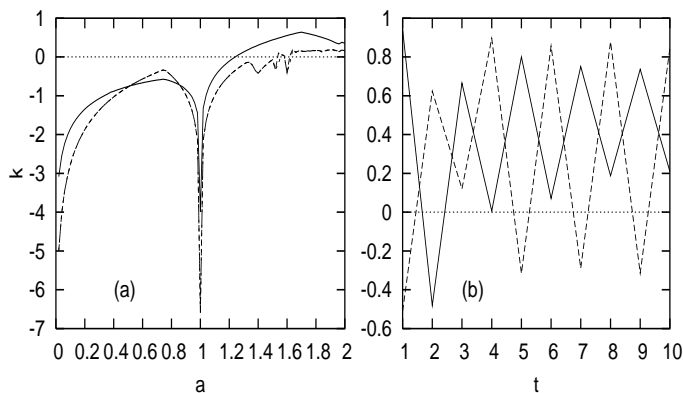


FIG. 2: (a) The transverse Liapunov exponent for the synchronization orbit of 1 and 2 (solid) in the coupled quadratic maps for the subsystem coupling. The dashed lines are the transverse Liapunov exponent between 1 and 3 for the subsystem and bidirectional coupling. (b) Time evolution of x_1 (solid) and x_2 (dashed) for parameter values where synchronization between 1 and 3 occurs. The parameter values are in (a) $k = 0.2$ and in (b) $k = 0.2$, $a = 1.4$. The initial conditions are the same as in Fig. 1.

respectively, for a system without delay (for subsystem coupling). When compared with the case with delay, we see that delay helps in the synchronization of 1 and 3 (since the transverse Liapunov is negative in a wider area) and it is detrimental for synchronization between 1 and 2.

We have studied the magnitude of the transverse Liapunov to find out which coupling, bidirectional or subsystem, results in faster synchronization. The more negative the transverse Liapunov is, the faster is the synchronization. As shown in Fig. 2(a) (dashed lines) the Liapunov exponent is essentially the same independently of the coupling type. In Fig. 2(b) we show the temporal evolution (for the subsystem coupling) of the orbit of element 1 and the orbit of element 2 in the region where 1 synchronizes with 3. We see that they are in antiphase synchronization, i.e., moving in opposite directions. Thus, 1 is in antiphase with 2, which in turn is in antiphase with 3 and consequently 1 is inphase with 3. So, this is the mechanism by which the relaying element, 2, helps in the synchronization of the outer elements, 1 and 3.

KURAMOTO OSCILLATORS

As in the case of discrete time system, we have found that the synchronization properties of the subsystem coupling are nearly identical to the synchronization properties of the bidirectional coupling. Because the subsystem configuration is simpler and allow us to perform some analytical studies in the Kuramoto system we will concentrate from now on mostly on the subsystem case.

The equations for the subsystem coupling in coupled Kuramoto oscillators are given by

$$\begin{aligned}\dot{\phi}_1 &= \omega - k \sin(\phi_1 - \phi_2^\tau) \\ \dot{\phi}_2 &= \omega' - k \sin(\phi_2 - \phi_1^\tau) \\ \dot{\phi}_3 &= \omega - k \sin(\phi_3 - \phi_2^\tau),\end{aligned}\quad (4)$$

where we have used the notation ϕ^τ to denote $\phi(t - \tau)$ and ϕ to denote $\phi(t)$. We see that oscillators 1 and 2 are uncoupled from oscillator 3 and in this way the analysis of two coupled oscillators is helpful in the understanding of the system as a whole. A system of two coupled Kuramoto oscillators with delay was studied by Schuster and Wagner[12] and is governed by

$$\begin{aligned}\dot{\phi}_1 &= \omega - k \sin(\phi_1 - \phi_2^\tau) \\ \dot{\phi}_2 &= \omega' - k \sin(\phi_2 - \phi_1^\tau)\end{aligned}\quad (5)$$

For the sake of completeness, here we reproduce the main results of [12]. They assumed that the synchronizing solution for the two coupled oscillators is given by

$$\phi_{1,2} = \psi \pm \alpha/2, \quad (6)$$

where ψ is the common dependent phase and α is a constant phase shift. Inserting this into Eq. 5 one gets

$$\cos(\psi - \psi^\tau) = (\omega - \omega')/(2k \sin(\alpha)) = \text{const}, \quad (7)$$

which is fulfilled if $\psi = \Omega t$. Thus, the synchronized solution is given by $\phi_{1,2} = \Omega t \pm \alpha/2$. Inserting this solution into Eq. 5 we get that the synchronizing frequency is given by the zeros of

$$f(\Omega) = \bar{\omega} - \Omega - k \tan(\Omega\tau) \sqrt{\cos^2(\Omega\tau) - k_c^2/k^2}, \quad (8)$$

where $k_c \equiv (\omega - \omega')/2$, $\bar{\omega} \equiv (\omega_1 + \omega_2)/2$. It is found that multiple solutions are possible for Ω when delay is present[12]. This is in contrast with the case without delay. The phase shift that belongs to the synchronized solution is

$$\begin{aligned}\alpha &= \arcsin(k_c/k \cos(\Omega\tau)), \quad \text{if } \cos(\Omega\tau) > 0, \\ \alpha &= \pi - \arcsin(k_c/k \cos(\Omega\tau)), \quad \text{otherwise.}\end{aligned}\quad (9)$$

If $\omega = \omega'$ the above equations will reduce to

$$f(\Omega) = \omega - \Omega - k \sin(\Omega\tau), \quad (10)$$

and

$$\begin{aligned}\alpha &= 0, \quad \text{if } \cos(\Omega\tau) > 0, \\ \alpha &= \pi. \quad \text{otherwise.}\end{aligned}\quad (11)$$

In Fig. 3(a) we show the region where synchronization between oscillator 1 and 2 occurs (white) and when it does not occur for a system with $\omega = \omega' = 1$. The phase separation in this case is π as shown above. The region

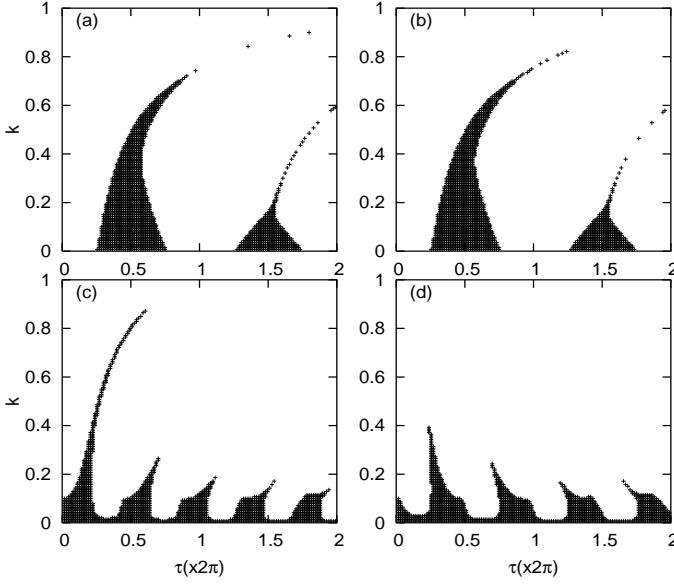


FIG. 3: Region where isochronal synchronization occurs (white) between oscillator 1 and 2 for the Kuramoto oscillators and when their phase separation is π (shaded) for (a) subsystem coupling and (b) bidirectional coupling, with $\omega = \omega' = 1$. Region where isochronal synchronization occurs (white) between oscillator 1 and 3 for the subsystem coupling with (c) $\omega = 1.2$, $\omega' = 1$, (d) $\omega = 1$, $\omega' = 1.2$. The initial conditions used were $\phi_1 = 0.1$, $\phi_2 = 0.12$, $\phi_3 = 0.14$ and $\phi_{1,2,3}(-\tau) = \phi_{1,2,3}(t=0) + 0.001\tau$.

of non synchronization is in the shape of "tongues" with its center being around $n\pi/2$, where n is an odd number.

We have found that when $\omega = \omega'$, oscillator 1 and 3 always synchronize, independently of the delay and coupling values. We can show this in the following way: give a small perturbation Δ , where Δ is positively defined, to the orbit of oscillator 1, so that the initial condition for oscillator 3 is given by $\phi_3 = \phi_1 + \Delta$. Now subtracting the first from the third lines of Eq. 4 we have

$$\dot{\phi}_3 - \dot{\phi}_1 = k[\sin(\phi_2^\tau - \phi_3) - \sin(\phi_2^\tau - \phi_1)], \quad (12)$$

Using Δ as defined above and the Taylor series expansion we get from Eq. 7 after a straight forward algebra,

$$\dot{\Delta} = -k\Delta \cos(\phi_2^\tau - \phi_1) = -k\Delta \cos(\Omega\tau - \alpha), \quad (13)$$

We have seen that if $\cos(\Omega\tau) > 0$, then $\alpha = 0$ and if $\cos(\Omega\tau) \leq 0$, then $\alpha = \pi$. Both cases result in $\dot{\Delta}/\Delta = -k|\cos(\Omega\tau)| < 0$. Consequently, the separation Δ between the orbits of oscillator 1 and 3 decreases as time evolves, implying that they tend to synchronize for any value of the parameter space (with $k > 0$). We also notice that there is region of the parameter space where the phase differences (say, 0 or π in the case of identical oscillators) do not change if the delay is varied in that region. Consequently, we expect that even if the delay between 1 and 2 is not exactly the same as between

2 and 3, zero-lag long range isochronal synchronization between 1 and 3 would still be possible.

For the sake of comparison, we show in Fig. 3(b) the region of synchronization between 1 and 2 (white) for the bidirectional coupling, that is,

$$\begin{aligned} \dot{\phi}_1 &= \omega - k \sin(\phi_1 - \phi_2^\tau) \\ \dot{\phi}_2 &= \omega' - k[\sin(\phi_2 - \phi_1^\tau) + \sin(\phi_2 - \phi_3^\tau)]/2 \\ \dot{\phi}_3 &= \omega - k \sin(\phi_3 - \phi_2^\tau). \end{aligned} \quad (14)$$

We see that Figs. 3 (a) and (b) are almost identical, confirming once more that there is basically no distinction between the synchronization properties of the subsystem and bidirectional couplings.

We have also considered the case in which $\omega \neq \omega'$. This is displayed in Figs. 3 (c) and (d). When $\omega \neq \omega'$ there is no isochronal synchronization between 1 and 2, since there is a constant phase difference between the oscillators[12]. However, synchronization between 1 and 3 is possible in a large area of the parameter space. The direction of the "tongues" where synchronization does not occur depend on having $\omega > \omega'$ or the other way around. However, the area of non synchronization varies little from one case to the other.

We have looked for even simpler configurations where the zero-lag long range isochronal synchronization would occur, such as removing one of the links between element 1 and 2 in the subsystem coupling, but we found that such a kind synchronization is not possible in those simpler configurations.

RÖSSLER OSCILLATORS

We have chosen Rössler chaotic oscillators to show that also in this chaotic system, governed by continuous time equations, the phenomenon reported in [6] is due to the fact that inphase synchronization coupled with another inphase synchronization results in isochronal synchronization. The equation of Rössler oscillator with subsystem coupling are given by

$$\begin{aligned} \dot{x}_i &= -\omega y_i - z_i - k(x_i - x_j^\tau) \\ \dot{y}_i &= \omega x_i + ay_i - k(y_i - y_j^\tau) \\ \dot{z}_i &= b + z(x_i - c) - k(z_i - z_j^\tau), \end{aligned} \quad (15)$$

where $i = 1, 2, 3$, and when $i = 1, j = 2$, $i = 2, j = 1$ and $i = 3, j = 2$. In this paper we have used, $\omega = 1$, $a = 0.15$, $b = 0.4$ and $c = 8.5$.

It is shown in Figs. 4(a) and (b) the region where synchronization happens (white) between oscillators 1 and 2 and between oscillators 1 and 3, respectively. We see once more that the presence of a middle element increases the region where synchronization is possible. In Fig. 4(c) we show the time evolution of $y_1(t)$ (solid) and $y_2(t)$

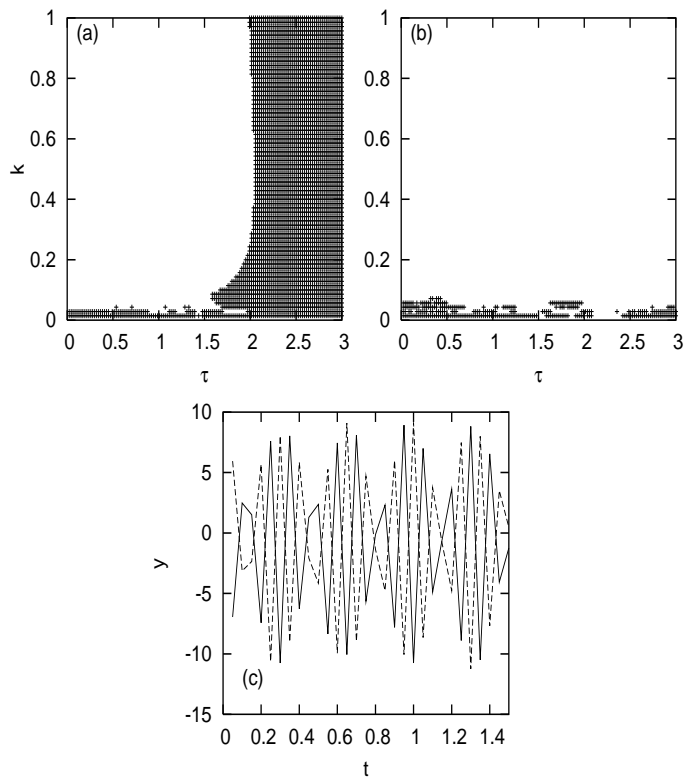


FIG. 4: Region where synchronization occurs (white) and when it does not (shaded) between (a) 1 and 2 and (b) 1 and 3 for the Rössler coupled oscillators with subsystem coupling. (c) Temporal evolution of $y_1(t)$ (solid) and $y_2(t)$ (dashed) when $\tau = 2.5$ and $k = 0.2$. The parameter values were $\omega = 1$, $a = 0.15$, $b = 0.4$ and $c = 8.5$, and the initial conditions where $x_1 = 0.1$, $y_1 = 0.12$, $z_1 = 0.14$, $x_2 = 0.1$, $y_2 = 0.22$, $z_2 = 0.24$, $x_3 = 0.1$, $y_3 = 0.12$, $z_3 = 0.14$ and $x, y, z(-\tau) = x, y, z(t = 0) + 0.001\tau$. (c) Temporal evolution of $x_1(t)$ (solid) and $x_2(t)$ (dashed) when $\tau = 2.5$ and $k = 0.2$.

(dashed) for a region where oscillator 1 and 3 synchronize, that is, $\tau = 2.5$ and $k = 0.2$. Once again we see the antiphase synchronization observed in the previous sections.

DISCUSSION

By studying the results presented in the paper by Fisher *et al.*, we see the origin of for zero-lag long range isochronal synchronization they observed is the same we

have found. If we take a look on Figs. 2(E), 2(F), 3(E) and 3(F) of [6] we see that they shifted the time series of the middle element by the delay value in order to facilitate the comparison with the time series of the outer elements. When they did this shift, the time series all the elements nearly matched each other. But it turns out that, apparently not realized by them, the value τ_c is the result of this antiphase mechanism mentioned in the previous paragraphs. In our paper we have clarified this point, and showed that there is a simpler configuration (the subsystem coupling) where nearly identical properties are observed when compared with the bidirectional coupling. We have also been able to perform analytical studies and show that for the Kuramoto system with identical oscillators, synchronization between the outer elements always occurs, independently of the parameter values.

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